# Tradeoffs between synchronization, communication, and work in parallel linear algebra computations

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# The problem, the algorithm, and the parallelization

- A **problem** asks for a solution which satisfies a set of properties with respect to a set of inputs (e.g. given $A$, find triangular matrices $L$ and $U$ such that $A = L \cdot U$)

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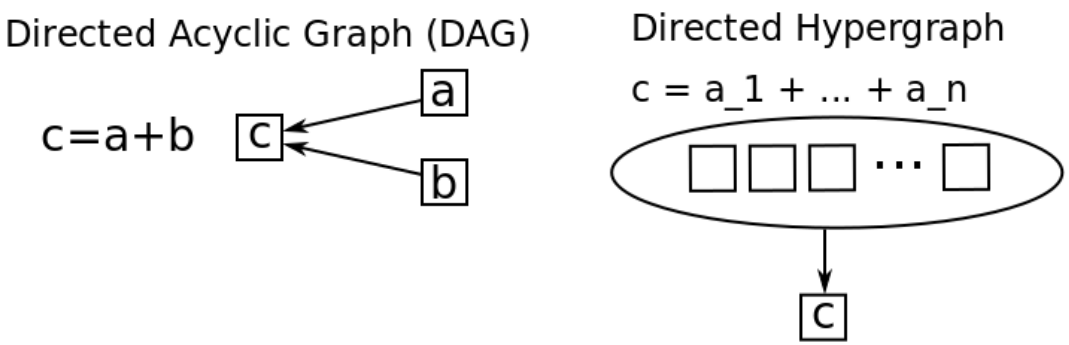
- An **algorithm** is a sequence of computer operations that determines a solution to the problem
- A **parallelization** is a schedule of the algorithm, which partitions the operations amongst processors and defines the necessary data movement between processors

# Graphical representation of a parallel algorithm

- We can represent an algorithm as a graph $G = (V, E)$ where
  - $V$ includes the input, intermediate, and output values used by the algorithm
  - $E$ represents the dependencies between pairs of values
  - e.g. to compute $c = a \cdot b$, we have $a, b, c \in V$ and $(a, c), (b, c) \in E$
- We can represent a set of algorithms by working with hypergraph representations $H = (V, \bar{E})$
  - $\bar{E}$ may represent the dependency of a value on a set of vertices (e.g. reduction tree)
  - e.g. to compute $d = \sum_{i=1}^{n} c_i$, we have $d, c_i \in V$ and hyperedges $(\{c_1, \ldots c_n\}, \{d\}) \in \bar{E}$



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  - $M_{ij}$ is the set of values sent by processor $i$ at timestep $j$

# Cost model

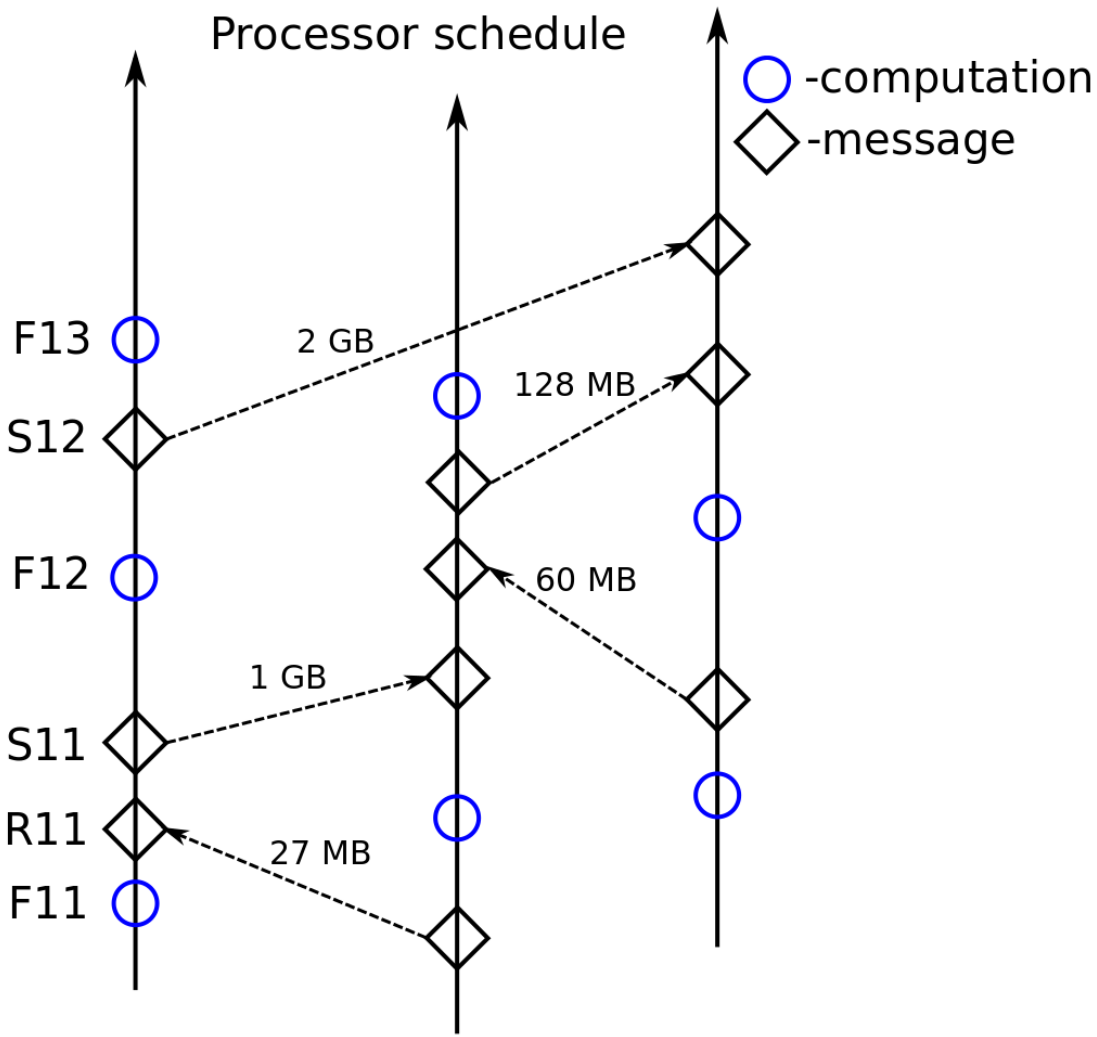
We quantify interprocessor communication and synchronization costs of a parallelization via a flat network model

- $\gamma$ - cost for a single computation (flop)
- $\beta$ - cost for a transfer of each byte between any pair of processors
- $\alpha$ - cost for a synchronization between any pair of processors

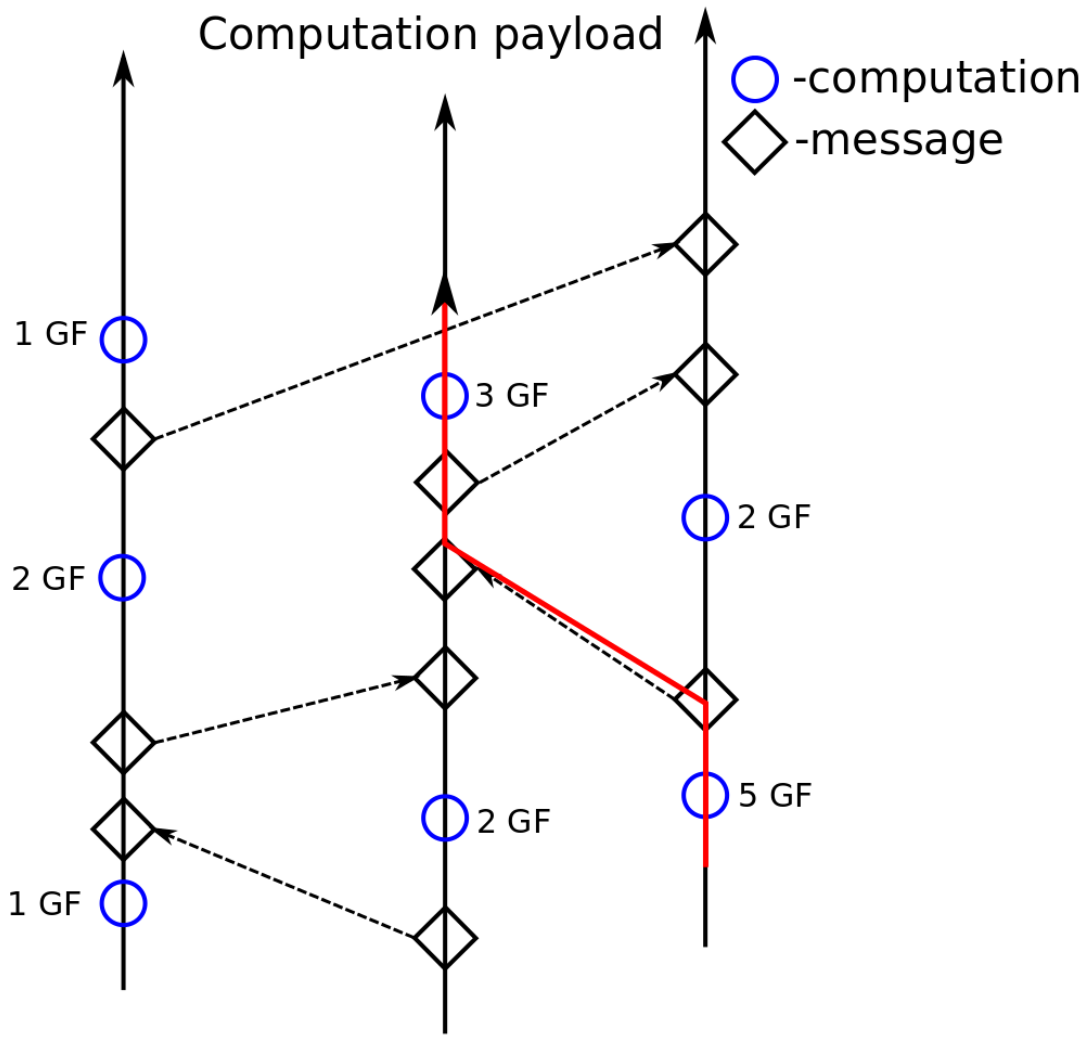
We measure the cost of a parallelization along the longest sequence of dependent computations and data transfers (critical path)

- $F$ - critical path payload for computation cost
- $W$ - critical path payload for communication (bandwidth) cost
- $S$ - critical path payload for synchronization cost

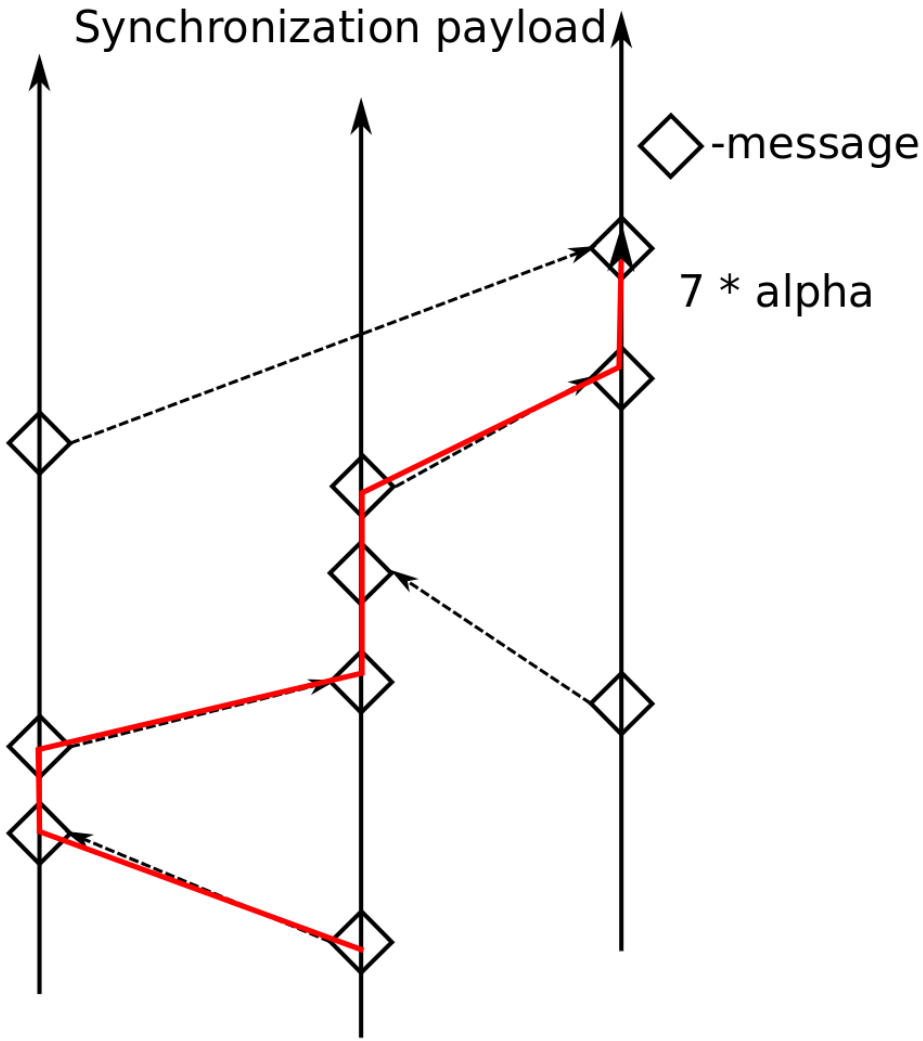
# Parallel schedule example



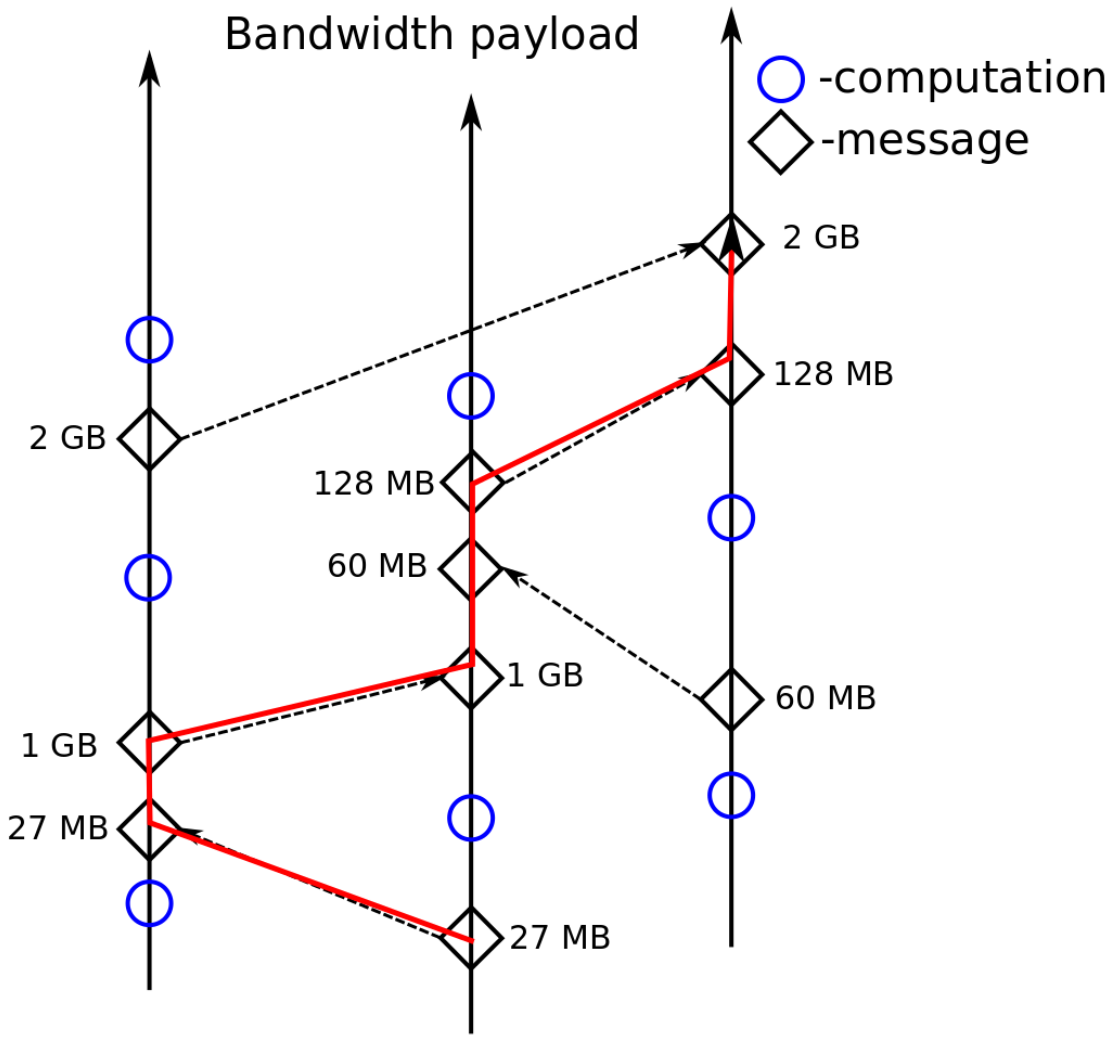
# Critical path for computational cost



# Critical path for synchronization cost



# Critical path for communication cost



# The duality of upper and lower bounds

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- A **parallelization** provides a schedule whose payload is an **upper bound** on the cost of the algorithm
- A parallelization of an algorithms optimal under a given cost model, if there exists a matching **lower bound** on the cost of the algorithm
- We will first give some parallel algorithms (upper bounds) and then show their optimality via a new technique
- The focus of our technique will be on lower bounds for synchronization cost, which manifest themselves as tradeoffs

# Solving a dense triangular system

**Problem**: For lower triangular dense matrix **L** and vector **y** of dimension $n$, solve

$$\mathbf{L} \cdot \mathbf{x} = \mathbf{y},$$

i.e., $\sum_{j=1}^{i} L_{ij} \cdot x_j = y_i$, for $i \in \{1, \ldots, n\}$.

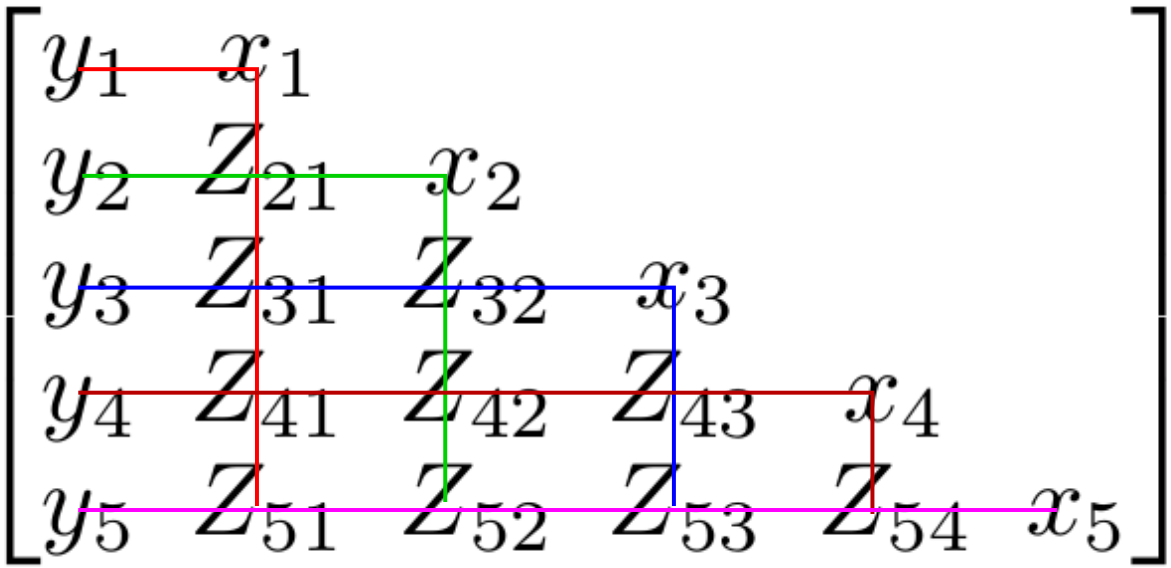
**Algorithm**:

$\mathbf{x} = \mathrm{TRSV}(\mathbf{L}, \mathbf{y}, n)$

1 **for** $i = 1$ **to** $n$
2 **for** $j = 1$ **to** $i - 1$
3 $Z_{ij} = L_{ij} \cdot x_j$
4 $x_i = \left(y_i - \sum_{j=1}^{i-1} Z_{ij}\right) / L_{ii}$

This algorithm does not immediately yield a graph representation since a summation order is not defined, but we can infer a hypergraph dependency representation

# Dependency Hypergraph: Triangular solve



# Schedules for triangular solve

One instance of the above triangular solve algorithm is a diamond DAG

- wavefront algorithms [Heath 1988]
- also algorithms given by [Papadimitriou and Ullman 1987] and [Tiskin 1998]

These Diamond DAG schedules achieve the following costs for $\mathbf{x}$ of dimension $n$, and some number of processors $p \in [1, n]$

- computational: $F_{\text{TRSV}} = O(n^2/p)$
- bandwidth: $W_{\text{TRSV}} = O(n)$
- synchronization: $S_{\text{TRSV}} = O(p)$

We see a tradeoff between computational and synchronization cost.

# Obtaining a Cholesky factorization

**Problem:** The Cholesky factorization of a symmetric positive definite matrix **A** is

$$\mathbf{A} = \mathbf{L} \cdot \mathbf{L}^T,$$

for a lower-triangular matrix **L**.

**Algorithm:**

$\mathbf{L} = \textsc{Cholesky}(\mathbf{A}, n)$

1 **for** $j = 1$ **to** $n$

2 $\quad L_{jj} = \sqrt{A_{ij} - \sum_{k=1}^{j-1} L_{jk} \cdot L_{jk}}$

3 $\quad$ **for** $i = j + 1$ **to** $n$

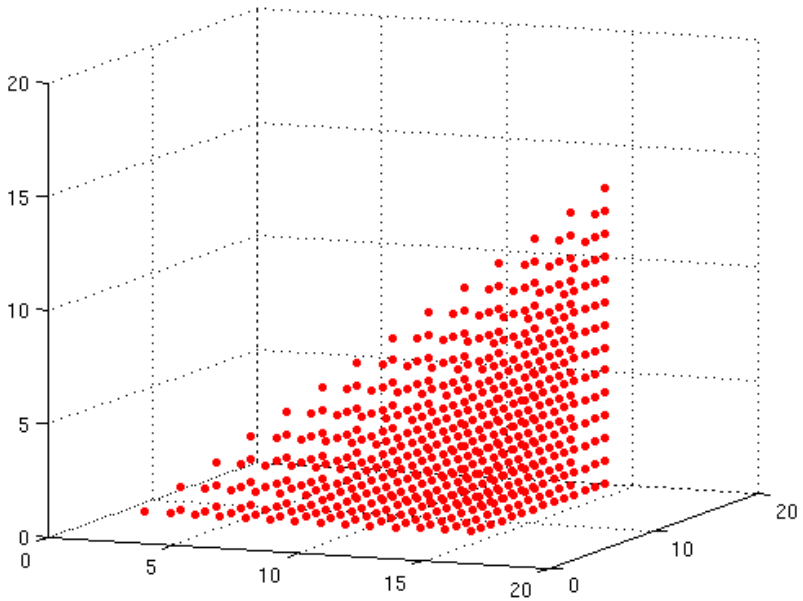
4 $\quad\quad$ **for** $k = 1$ **to** $j - 1$

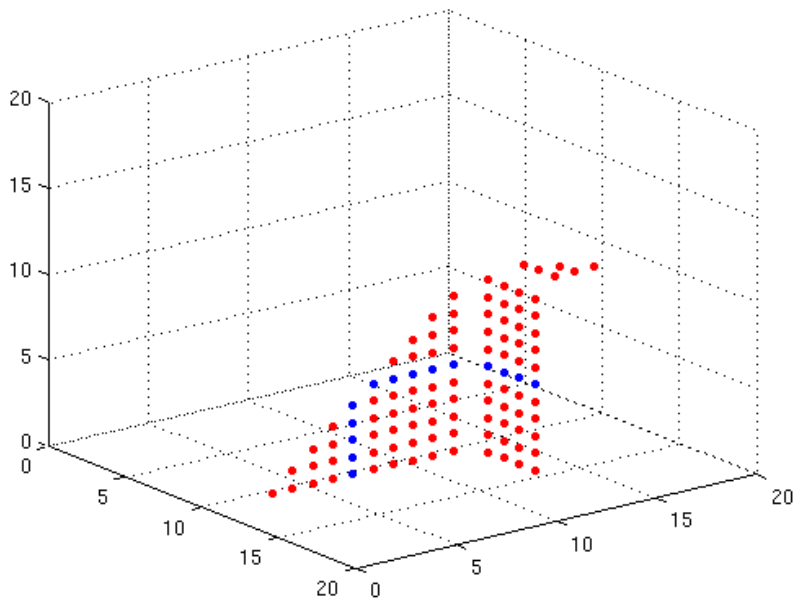
5 $\quad\quad\quad Z_{ijk} = L_{ik} \cdot L_{jk}$

6 $\quad\quad L_{ij} = (A_{ij} - \sum_{k=1}^{j-1} Z_{ijk})/L_{jj}$

Again this algorithm yields only a hypergraph dependency representation.

# Cholesky dependency hypergraph




These diagrams show (a) the vertices $Z_{ijk}$ in $V_{\mathrm{GE}}$ with $n = 16$ and (b) the hyperplane $x_{12}$ and hyperedge $e_{12,6}$ on $H_{\mathrm{GE}}$.

# Parallelizations of Cholesky

For **A** of dimension $n$, with $p$ processors, and some $c \in [1, p^{1/3}]$ [Tiskin 2002] in BSP and 2.5D algorithms [ES, JD 2011] achieve the costs

- computational: $F_{\mathrm{GE}} = O(n^3/p)$
- bandwidth: $W_{\mathrm{GE}} = O(n^2/\sqrt{cp})$
- synchronization: $S_{\mathrm{GE}} = O(\sqrt{cp})$

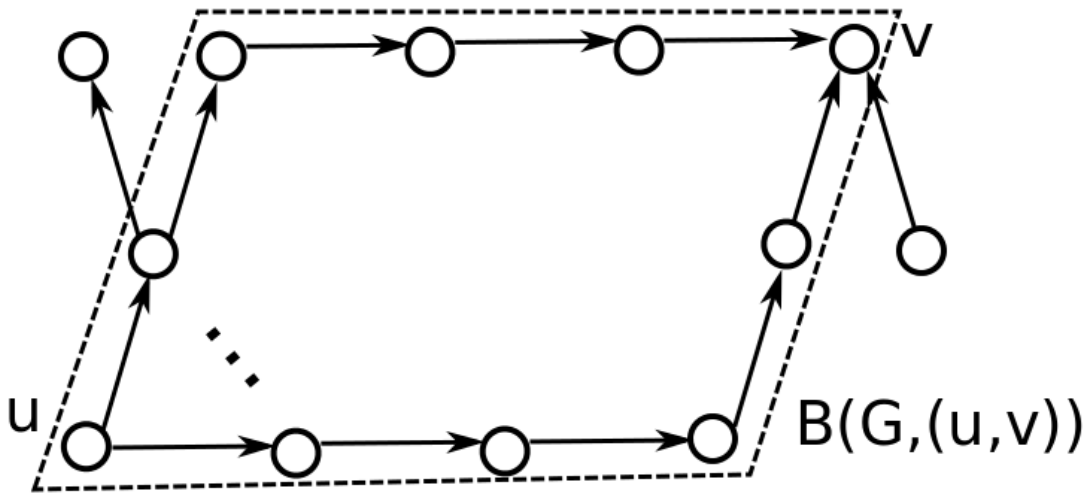
We see a tradeoff between computational and synchronization cost, as well as a tradeoff between bandwidth and synchronization costs (the latter independent of the number of processors, $p$).

Algorithms with the same asymptotic costs also exist for LU with pairwise or with tournament pivoting as well as for QR factorization

# Dependency bubble

## Definition (Dependency bubble)

Given two vertices $u, v$ in a directed acyclic graph $G = (V, E)$, the dependency bubble $B(G, (u, v))$ is the union of all paths in $G$ from $u$ to $v$.



# Path-expander graph

**Definition ($(\epsilon, \sigma)$-path-expander)**

Graph $G = (V, E)$ is a $(\epsilon, \sigma)$-**path-expander** if there exists a path $(u_1, \ldots u_n) \subset V$, such that the dependency bubble $B(G, (u_i, u_{i+b}))$ has size $\Theta(\sigma(b))$ and a minimum cut of size $\Omega(\epsilon(b))$.

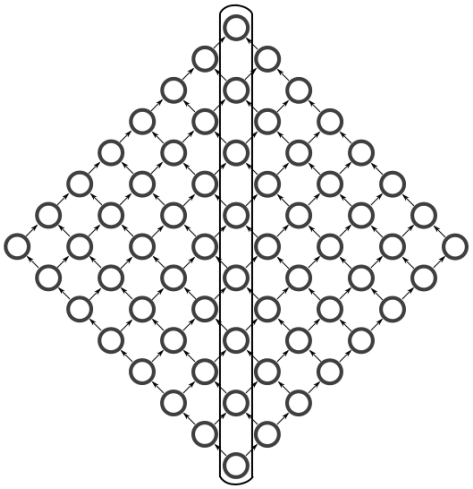
# Scheduling tradeoffs of path-expander graphs

**Theorem (Path-expander communication lower bound)**

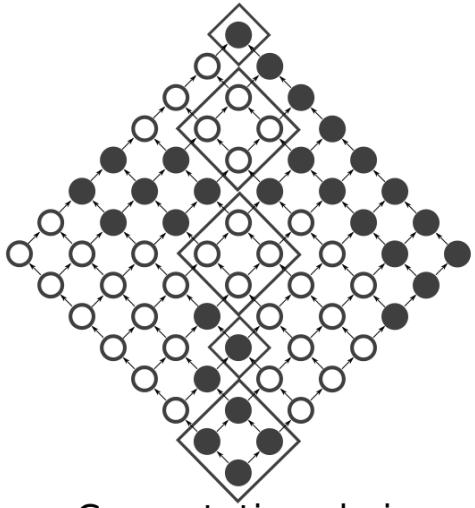
*Any parallel schedule of an algorithm, with a* $(\epsilon, \sigma)$-**path-expander** *dependency graph about a path of length* $n$ *incurs the computation* $(F)$*, bandwidth* $(W)$*, and latency* $(S)$ *costs for some* $b \in [1, n]$*,*

$$F = \Omega\left(\sigma(b) \cdot n/b\right), \quad W = \Omega\left(\epsilon(b) \cdot n/b\right), \quad S = \Omega\left(n/b\right).$$

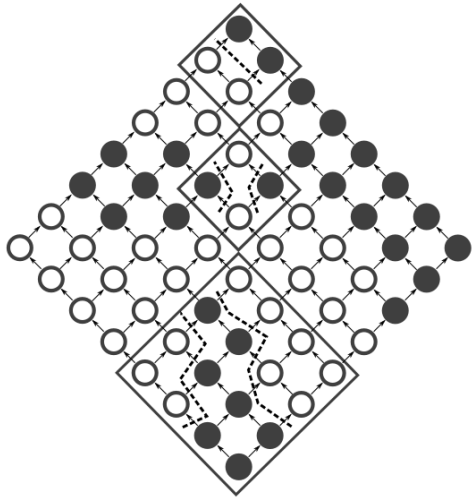
# Demonstration of proof for $(b, b^2)$-path-expander



Dependency path P



Computation chain



Communication chain

# Tradeoffs for triangular solve

**Theorem**

*Any parallelization of any dependency graph $G_{\mathrm{TRSV}}(n)$ incurs the following computation ($F$), bandwidth ($W$), and latency ($S$) costs, for some $b \in [1, n]$,*

$$F_{\mathrm{TRSV}} = \Omega\left(n \cdot b\right), \qquad W_{\mathrm{TRSV}} = \Omega\left(n\right), \qquad S_{\mathrm{TRSV}} = \Omega\left(n/b\right),$$

*and furthermore, $F_{\mathrm{TRSV}} \cdot S_{\mathrm{TRSV}} = \Omega\left(n^2\right)$.*

**Proof.**

Proof by application of path-based tradeoffs since $G_{\mathrm{TRSV}}(n)$ is a $(b, b^2)$-**path-expander** dependency graph. □

# Attainability and related work on triangular solve

Diamond DAG lower bounds were also given by

- Papadimitriou and Ullman [P.U. 1987]
- Tiskin [T. 1998]

Previously noted algorithms for triangular solve attain these lower bounds with the number of processors $p = n/b$

- computational: $F_{\mathrm{TRSV}} = \Theta(n^2/p)$
- bandwidth: $W_{\mathrm{TRSV}} = \Theta(n)$
- synchronization: $S_{\mathrm{TRSV}} = \Theta(p)$

# Tradeoffs for Cholesky

**Theorem**

*Any parallelization of any dependency graph* $G_{\mathrm{GE}}(n)$ *incurs the following computation (*$F$*), bandwidth (*$W$*), and latency (*$S$*) costs, for some* $b \in [1, n]$,

$$F_{\mathrm{GE}} = \Omega\left(n \cdot b^2\right), \qquad W_{\mathrm{GE}} = \Omega\left(n \cdot b\right), \qquad S_{\mathrm{GE}} = \Omega\left(n/b\right),$$

*and furthermore,* $F_{\mathrm{GE}} \cdot S_{\mathrm{GE}}^2 = \Omega\left(n^3\right), \quad W_{\mathrm{GE}} \cdot S_{\mathrm{GE}} = \Omega\left(n^2\right).$

**Proof.**

Proof by showing that $G_{\mathrm{GE}}(n)$ is a $(b^2, b^3)$-**path-expander** about the path corresponding to the calculation of the diagonal elements of **L**.

# Parallelizations of Cholesky

These lower bounds are attainable for $b \in [n/\sqrt{p}, n/p^{2/3}]$ by existing algorithms, so for $c \in [1, p^{1/3}]$,

- computational: $F_{\mathrm{GE}} = \Theta(n^3/p)$
- bandwidth: $W_{\mathrm{GE}} = \Theta(n^2/\sqrt{cp})$
- synchronization: $S_{\mathrm{GE}} = \Theta(\sqrt{cp})$

Therefore, we have a tradeoff between communication and synchronization

$$W_{\mathrm{GE}} \cdot S_{\mathrm{GE}} = \Theta(n^2)$$

We conjecture that our lower bound technique is extensible to QR and symmetric eigensolve algorithms.

# Krylov subspace methods

We consider the $s$-step Krylov subspace basis computation

$$\mathbf{x}^{(l)} = \mathbf{A} \cdot \mathbf{x}^{(l-1)},$$

for $l \in \{1, \ldots, s\}$ where the graph of the symmetric sparse matrix $\mathbf{A}$ is a $(2m+1)^d$-point stencil.

# Cost tradeoffs for Krylov subspace methods on stencils

**Theorem**

*Any parallel execution of an s-step Krylov subspace basis computation for a $(2m+1)^d$-point stencil, requires the following computational, bandwidth, and latency costs for some $b \in \{1, \ldots s\}$,*

$$F_{\mathrm{Kr}} = \Omega\left(m^d \cdot b^d \cdot s\right), W_{\mathrm{Kr}} = \Omega\left(m^d \cdot b^{d-1} \cdot s\right), S_{\mathrm{Kr}} = \Omega\left(s/b\right).$$

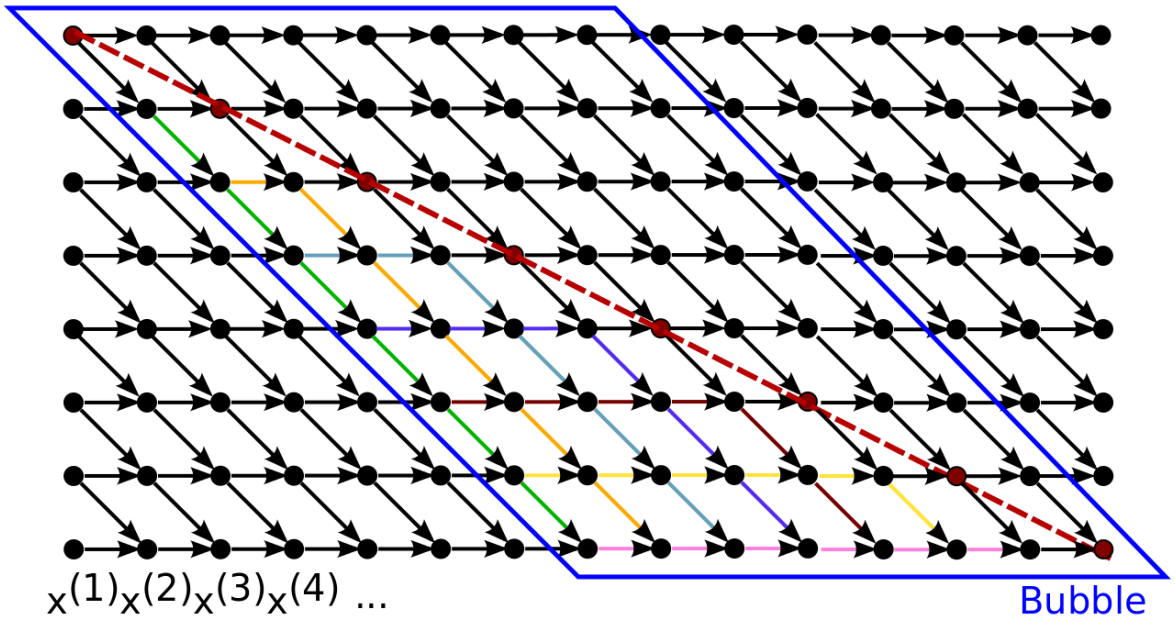
*and furthermore,*

$$F_{\mathrm{Kr}} \cdot S_{\mathrm{Kr}}^d = \Omega\left(m^d \cdot s^{d+1}\right), \quad W_{\mathrm{Kr}} \cdot S_{\mathrm{Kr}}^{d-1} = \Omega\left(m^d \cdot s^d\right).$$

# Proof of tradeoffs for Krylov subspace methods

**Proof.**

Done by showing that the dependency graph of a $s$-step $(2m+1)^d$-point stencil is a $(m^d b^d, m^d b^{d+1})$-**path-expander**. □



# Attainability

The lower bounds may be attained via communication-avoiding $s$-step algorithms (PA1 in Demmel, Hoemmen, Mohiyuddin, and Yelick 2007)

$$F_{\mathrm{Kr}} = O\left(m^d \cdot b^d \cdot s\right), W_{\mathrm{Kr}} = O\left(m^d \cdot b^{d-1} \cdot s\right), S_{\mathrm{Kr}} = O\left(s/b\right),$$

under the assumption $n/p^{1/d} = O(bm)$.

# Counterexample: All-Pairs Shortest-Paths (APSP)

The APSP problem seeks to find the shortest paths between all pairs of vertices in graph $G$

The Floyd-Warshall algorithm is analogous to Gaussian elimination on a different semiring and achieves costs

$$F_{\mathrm{FW}} = \Theta(n^3/p) \quad W_{\mathrm{FW}} = \Theta(n^2/\sqrt{cp}) \quad \mathbf{S_{FW} = \Theta(\sqrt{cp})}$$

Alexander Tiskin demonstrated that it is possible to augment path-doubling to achieve the costs

$$F_{\mathrm{APSP}} = O(n^3/p) \quad W_{\mathrm{APSP}} = O(n^2/\sqrt{cp}) \quad \mathbf{S_{APSP} = O(\log p)}$$

# Summary and conclusion

Proved tight synchronization lower bounds on schedules for algorithms

- triangular solve $F_{\mathrm{TRSV}} \cdot S_{\mathrm{TRSV}} = \Omega\left(n^2\right)$
- Gaussian Elimination
  $F_{\mathrm{GE}} \cdot S_{\mathrm{GE}}^2 = \Omega\left(n^3\right)$ and $W_{\mathrm{GE}} \cdot S_{\mathrm{GE}} = \Omega(n^2)$
- $s$-step Krylov subspace methods on $(2m+1)^d$-pt stencils
  $F_{\mathrm{Kr}} \cdot S_{\mathrm{Kr}}^d = \Omega\left(m^d \cdot s^{d+1}\right)$ and $W_{\mathrm{Kr}} \cdot S_{\mathrm{Kr}}^{d-1} = \Omega\left(m^d \cdot s^d\right)$

Our lower bounds also extend naturally to a number of graph algorithms

- Floyd-Warshall is analogous to Gaussian Elimination
- Bellman-Ford is analogous to Krylov subspace methods

However, it may be possible to find **new algorithms** rather than **parallel schedules** to solve these **problems** with less synchronization and communication cost (e.g. Tiskin's APSP algorithm, 2001)