CS 598: Communication Cost Analysis of Algorithms
Lecture 15: Communication-optimal sorting and tree-based algorithms

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Communication-optimal sorting

The best complexity achievable by a parallel comparison-based sort is

$$T_{\text{sort}}(n, P, H) = \Omega\Big(\frac{n\log_2(n)}{P}\cdot\gamma + \frac{n\log_2(n)}{P\log_2(\sqrt{n/P})}\cdot\beta + \frac{\log_2(n)}{\log_2(\sqrt{n/P})}\cdot\alpha\Big)$$

recall $\log_2(n)/\log_2(\sqrt{n/P}) = \log_{\sqrt{n/P}}(n)$

- R. Cole's parallel mergesort (1988) achieves this for $P = \Theta(n)$
- M.T. Goodrich (1999) provides a mixed mergesort/samplesort that attains the cost for arbitrary P
- Cole's construction uses a pipelined binary tree, Goodrich extends it to an s-ary tree with $s = \sqrt{n/P}$

Sampling and merging samples

The parallel mergesort algorithm is defined in a bottom-up way

- we first place and sort subsequences of size $s^2 = n/P$ on P leaves of the s-ary tree
- for each tree node u, we will merge the s subsequences assigned to its children $\pi(u)$

$$L(u) = \text{sort}\left(\bigcup_{v \in \pi(u)} L(v) \right)$$

- each tree node will distribute its subsequence over s more processors than those of its children
- since the tree is of height $\log_s(P)$, the root node will end with n sorted elements distributed over P processors
- at tree node with height h will merge s subsequences of size s^h, using a regular sample of total size s^h
- Q: sounds simple enough... so why do we need to pipeline?
- A: merging the regular subsamples is almost as hard as the merge itself, so we will construct them gradually

The short life of an exponentially-growing tree node

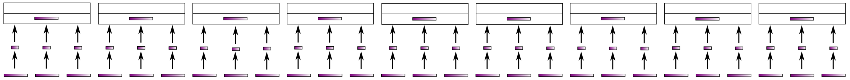
Nodes in the s-ary tree are either *waiting*, *growing*, or *full*

- node u at the start of iteration t:
    - has a subset of its target subsequence $L_t(u) \subseteq L(u)$
    - is *waiting* if $L_t(u) = \emptyset$
    - is *growing* if $L_t(u) \neq \emptyset$ and $L_t(u) \neq L(u)$
    - is *full* if it has its target subsequence $L_t(u) = L(u)$
- leaves are full at iteration 0
- each tree node u waits up to iteration t, the first iteration at which its children have a subsequence of size s^2, then collects a sample of size s from each and merges sequentially, so $|L_t(u)| = s^2$
- if u is growing at iteration t, then it uses $L_t(u)$ as a sample to merge

  $$L_{t+1}(u) = \text{sort}\left(\bigcup_{v \in \pi(u)} \mathcal{R}(|L_t(u)|, L_t(v)) \right)$$

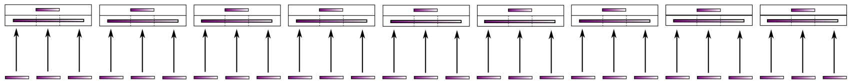
  where $\mathcal{R}(k, U)$ extracts a regular sample of size k from U
- growing nodes grow by a factor of s, i.e. $|L_{t+1}(u)| = s|L_t(u)|$

Communication-optimal sort [Cole 1988], [Goodrich 1999]



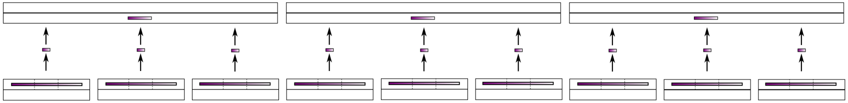
1. Pass samples up and merge

Communication-optimal sort [Cole 1988], [Goodrich 1999]



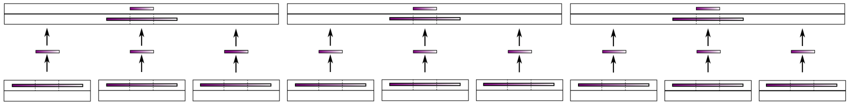
3. Merge subpartitions

Communication-optimal sort [Cole 1988], [Goodrich 1999]



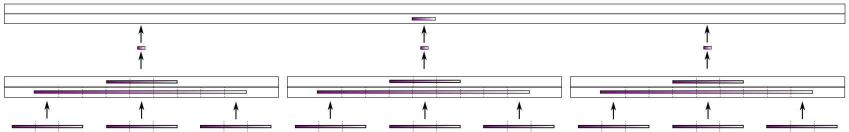
4. Pass up new samples

Communication-optimal sort [Cole 1988], [Goodrich 1999]



5. Pass up larger samples, use old sample to split

Communication-optimal sort [Cole 1988], [Goodrich 1999]



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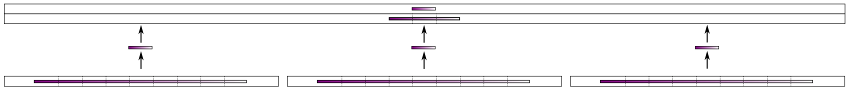

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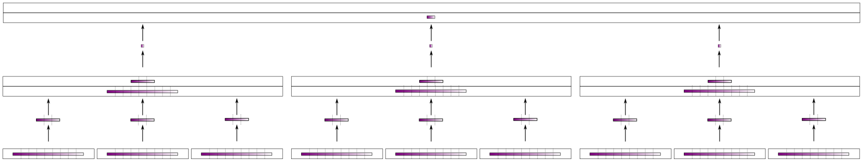
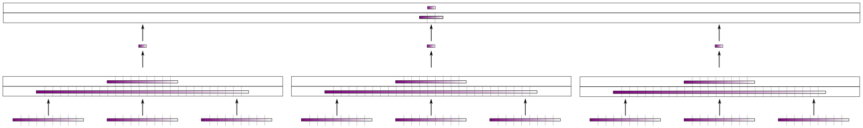


zoom out horizontally

Communication-optimal sort [Cole 1988], [Goodrich 1999]



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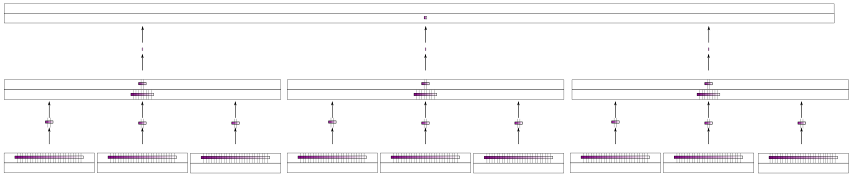
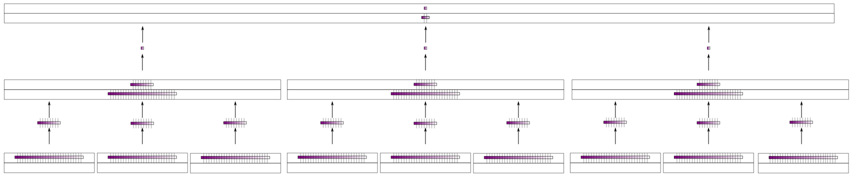


Zoom out again!

Communication-optimal sort [Cole 1988], [Goodrich 1999]



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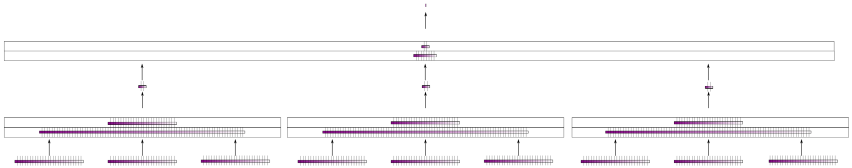

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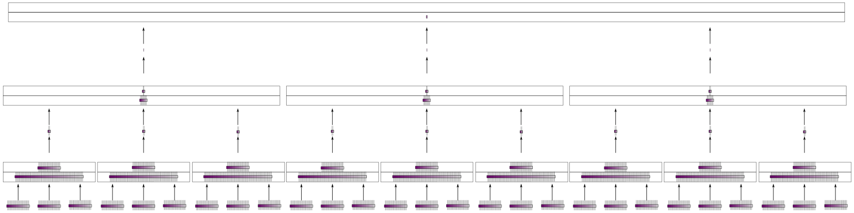
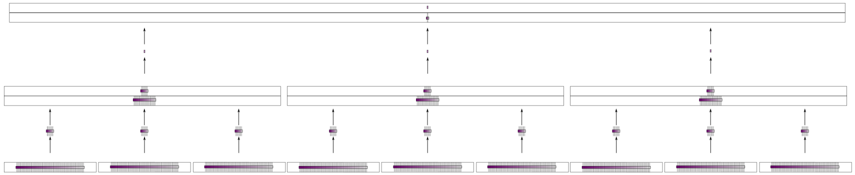


Zoom out again! Now sorting with 3^6=729 processors!

Communication-optimal sort [Cole 1988], [Goodrich 1999]



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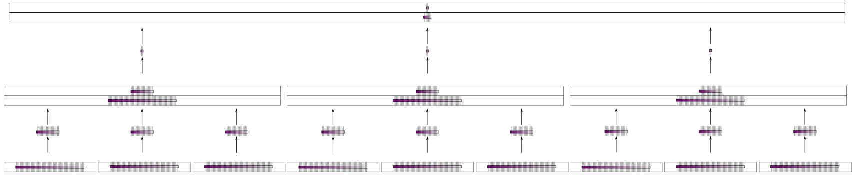

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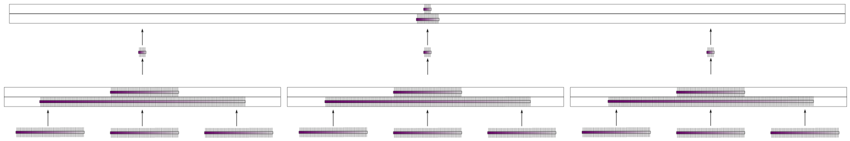

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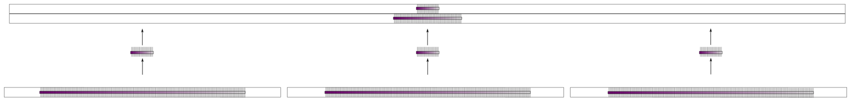

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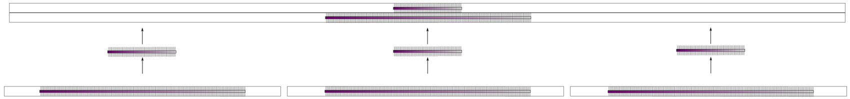

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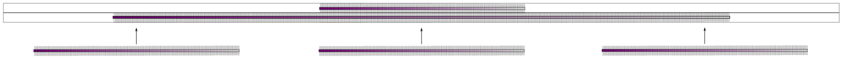

Sampling guarantees

At every step of the algorithm, we merge s sequences of size s^t using a sample of size s^t, with s^{t-1} processors

- we show can partition the sequences such that each processor gets no more than $2s^2 = 2n/P$ elements
- there are at most s elements in a sequence between two consecutive sample elements selected from that sequence
    - this is nontrivial, but holds as a consequence of a regular sample of s^t elements being a regular sample of a regular sample of s^{t+1} elements
- we select $s^{t-1} - 1$ splitters from sorted sample of size s^t
- if there are k_j elements from subsequence j between splitter i and $i+1$, it has at most $(k_j + 1)s$ elements between these splitters
- we can bound total $\sum_{j=1}^{s} k_j = s$
- the total number of elements in each interval is bounded by $\sum_{j=1}^{s}(k_j + 1)s \leq 2s^2$

Cost analysis of Cole/Goodrich parallel mergesort

There are $\log_s(P)$ levels in the tree

- a parent become full 3 iterations after its children become full, so the algorithm terminates in $3\log_s(P)$ iterations
- the amount of work done at each step for growing parent is a factor of s less than the child, and there are s less nodes at the tree level
- therefore, the work and memory usage decrease geometrically up from the highest level that is full
- processing every node requires $O(s^2 \cdot \beta)$ and as much memory from every processor involved
- therefore, each iteration can be executed with $O(s^2 \cdot \beta)$ communication and as much memory

$$W_{\text{sort}}(n,P) = O(s^2 \log_s(n) \cdot \beta) = O(n \log_{\sqrt{n/P}}(n)/P \cdot \beta)$$

Short pause

The Parallel Random Access Machine (PRAM)

The PRAM model is perhaps the most popular traditional parallelism model and is still in use today

- use a maximal number of processors suitable to compute the algorithm in a minimal number of parallel steps (depth)
- e.g. Cole's mergesort uses $O(n)$ processors to sort in $O(\log(n))$ steps
- all processors can access a global memory
    - EREW (exclusive read exclusive write) - processors cannot access the same memory locations in a parallel step
    - CREW (concurrent read exclusive write) - processors can read but not write to the same memory locations in a parallel step
    - CRCW (concurrent read concurrent write) - processors can read and write to the same memory locations in a parallel step
    - for CRCW model need to define how writes are arbitrated (common, random, priority)
- Brent's Lemma: an EREW PRAM algorithm that uses P processors and s steps can be computed with in P/k processors in sk steps

Tree contraction

Consider an algebraic or boolean expression, e.g.

$$T = \Big((a+b)\cdot(c+d)\Big)\cdot\Big((e+f+g)\cdot(h+i)\Big)\cdot(j+k)$$

- the problem can be represented as a rooted tree
- computing T corresponds to *tree contraction*
- we can use tree contraction for important tasks such as tree isomorphism and subexpression elimination in dataflow analysis
- a naive algorithm would recursively evaluate the n nodes in the tree, starting with the leaves
- Q: how many parallel steps would such an algorithm require in the worst case?
- A: $O(n)$ if the tree has height $O(n)$

Parallel tree contraction

Miller and Reif (1989) provide a famous PRAM algorithm for the problem

- their algorithm consists of *rake* and *compress* steps
- *rake* evaluates all the leaves of the tree
- *compress* contracts all chains in the tree
    - a chain is any connected subtree where each node has only one child
    - compress removes every other node in the chain
- Q: can you see why $O(\log(n))$ rake and compress steps would contract the tree?
- A: any new tree branch increases the number of leaves by one

How to compress in parallel?

It is nontrivial to identify which nodes in a chain are odd or even

- Miller and Reif provide deterministic and randomized solutions
- the deterministic solution splits chains using pointer chasing
  - we start with every child pointing to its parent
  - if parent has one child, point to grandparent (if exists)
  - this splits each chain into two, in one of the chains, a node that participated in compress now has no child
  - we proceed ensuring that this node is never evaluated
  - PRAM requires $O(n)$ processors and $O(\log(n))$ steps
- the randomized solution deletes a subset of nodes in any chain
  - randomly assign 1 or 0 to each node in the chain
  - pointer chase from every node marked with 0 whose parent and child are marked with 1
  - if s nodes are part of chains, we delete $\Omega(s)$ nodes without breaking chains
  - PRAM requires $O(n/\log(n))$ processors and $O(\log(n))$ steps

Randomized Miller and Reif algorithm in BSP

So, how do we do tree contraction in the BSP model?

- perform $O(s)$ accesses and pointer chases needed in a PRAM step using $O(1)$ BSP supersteps and $O(s/P)$ communication
- with each step of rake/compress we decrease the number of nodes (s) geometrically
- need to assume the accesses/nodes are load balanced (can randomly permute initially)
- the communication cost then goes down geometrically
- after $O(\log(P))$ steps, the size of the tree is $O(n/P)$, so we can collect all nodes on one processor and contract the tree locally
- the total cost is then

$$O(n/P \cdot \beta + \log(P) \cdot \alpha)$$