Tradeoffs between synchronization, communication, and work in parallel schedules

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Outline

Graphical representation of a computation

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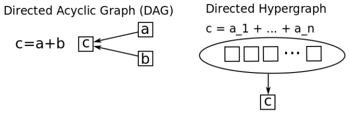

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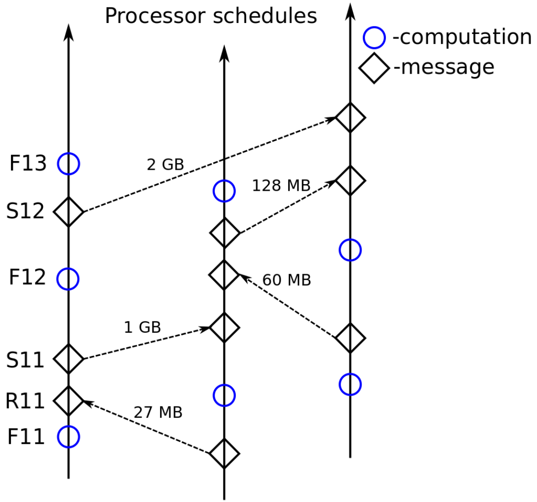
  - M_{ij} is the set of values sent by processor i at timestep j

Parallel schedule example



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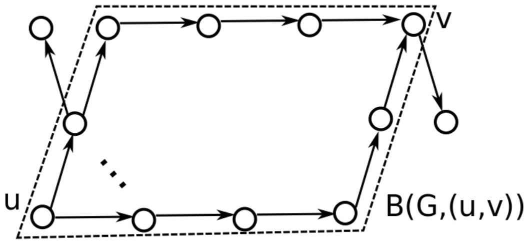
- For all non-local dependency paths in G, there must exist a sequence of messages in the schedule

Dependency bubble

Definition (Dependency bubble)

Given two vertices u, v in a directed acyclic graph $G = (V, E)$, the dependency bubble $B(G, (u, v))$ is the union of all paths in G from u to v.



Path-expander graph

Definition ((ϵ, σ)-path-expander)

Graph $G = (V, E)$ is a (ϵ, σ)-**path-expander** if there exists a path $(u_1, \dots u_n) \subset V$, such that the dependency bubble $B(G, (u_i, u_{i+b}))$ has size $\Omega(\sigma(b))$ and a minimum cut of size $\Omega(\epsilon(b))$.

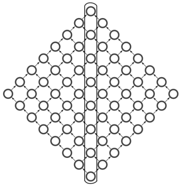
Scheduling tradeoffs of path-expander graphs

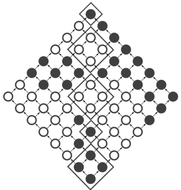
Theorem (Path-expander communication lower bound)

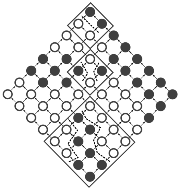
Any parallel schedule of an algorithm, with a (ϵ, σ)-**path-expander** *dependency graph* $G = (V, E)$ *about a path of length* n *incurs the computation* (F)*, bandwidth* (W)*, and latency* (S) *costs*

$$F = \Omega\left(\sigma(b) \cdot n/b\right), \quad W = \Omega\left(\epsilon(b) \cdot n/b\right), \quad S = \Omega\left(n/b\right).$$

An example (b, b^2)-**path-expander**


Dependency path P


Computation chain


Communication chain

Application: Triangular solve

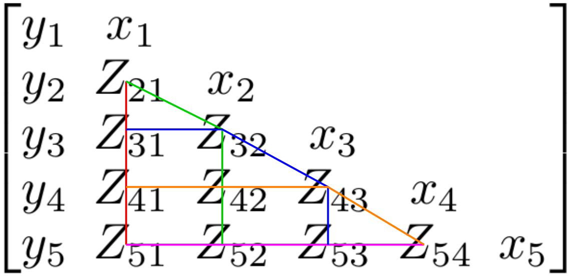
For lower triangular dense $\mathbf{L}$, solve

$$\mathbf{L} \cdot \mathbf{x} = \mathbf{y},$$

i.e., $\sum_{j=1}^{i} L_{ij} \cdot x_j = y_i$, for $i \in \{1, \ldots, n\}$.

```
$\mathbf{x} = \mathrm{TRSV}(\mathbf{L}, \mathbf{y}, n)$
1  for $i = 1$ to $n$
2      for $j = 1$ to $i - 1$
3          $Z_{ij} = L_{ij} \cdot x_j$
4      $x_i = \left(y_i - \sum_{j=1}^{i-1} Z_{ij}\right) / L_{ii}$
```

Dependency Hypergraph: Triangular solve



$$\begin{bmatrix} y_1 & x_1 & & & & \\ y_2 & Z_{21} & x_2 & & & \\ y_3 & Z_{31} & Z_{32} & x_3 & & \\ y_4 & Z_{41} & Z_{42} & Z_{43} & x_4 & \\ y_5 & Z_{51} & Z_{52} & Z_{53} & Z_{54} & x_5 \end{bmatrix}$$

Bandwidth lower bound for TRSV

Theorem

Any parallelization of any dependency graph $G_{\mathrm{TRSV}}(n)$ where two processors compute $\lfloor n^2/p \rfloor$ elements of $\mathbf{Z}$ *must incur a communication cost of*

$$W_{\mathrm{TRSV}} = \Omega\left(n/\sqrt{p}\right).$$

Proof.

Proof by application of lower bound on 2D lattice Hypergraph cut. $\square$

Tradeoffs for TRSV

Theorem

Any parallelization of any dependency graph $G_{\mathrm{TRSV}}(n)$ *incurs the following computation (F), bandwidth (W), and latency (S) costs, for some* $b \in [1, n]$,

$$F_{\mathrm{TRSV}} = \Omega\left(n \cdot b\right), \qquad W_{\mathrm{TRSV}} = \Omega\left(n\right), \qquad S_{\mathrm{TRSV}} = \Omega\left(n/b\right),$$

and furthermore, $F_{\mathrm{TRSV}} \cdot S_{\mathrm{TRSV}} = \Omega\left(n^2\right)$.

Proof.

Proof by application of path-based tradeoffs since $G_{\mathrm{TRSV}}(n)$ is a (b, b^2)-**path-expander** dependency graph. □

Attainability and related work on TRSV

Diamond DAG lower bounds were also given by

- Papadimitriou and Ullman [P.U. 1987]
- Tiskin [T. 1998]

Efficient algorithms for TRSV attain above lower bounds

- wavefront algorithms (Heath 1988)
- also algorithms given by [P.U 1987] and [T. 1998]

Application: Cholesky factorization

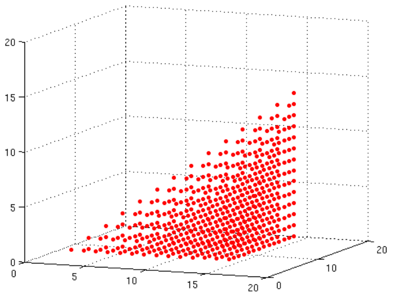
The Cholesky factorization of a symmetric positive definite matrix $\mathbf{A}$ is

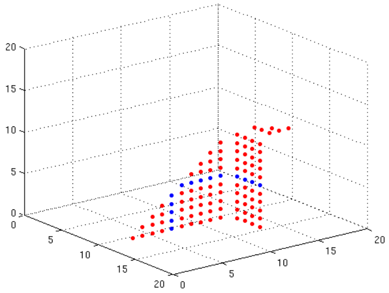
$$\mathbf{A} = \mathbf{L} \cdot \mathbf{L}^T,$$

for a lower-triangular matrix $\mathbf{L}$.

```
$\mathbf{L} = \textsc{Cholesky}(\mathbf{A}, n)$
1  for $j = 1$ to $n$
2      $L_{jj} = \sqrt{A_{ij} - \sum_{k=1}^{j-1} L_{jk} \cdot L_{jk}}$
3      for $i = j + 1$ to $n$
4          for $k = 1$ to $j - 1$
5              $Z_{ijk} = L_{ik} \cdot L_{jk}$
6          $L_{ij} = (A_{ij} - \sum_{k=1}^{j-1} Z_{ijk})/L_{jj}$
```

Cholesky dependency hypergraph





These diagrams show (a) the vertices Z_{ijk} in V_{GE} with $n = 16$ and (b) the hyperplane x_{12} and hyperedge $e_{12,6}$ on H_{GE}.

Cholesky bandwidth cost lower bound

Theorem

Any p-processor parallelization of the dependency graph $G_{\mathrm{GE}}(n)$ *must incur a communication of*

$$W_{\mathrm{GE}} = \Omega\left(n^2/p^{2/3}\right).$$

Proof.

Employs 3D lattice hypergraph cut lower bound and assumes some work balance. □

Tradeoffs for Cholesky

Theorem

Any parallelization of any dependency graph $G_{\mathrm{GE}}(n)$ incurs the following computation (F), bandwidth (W), and latency (S) costs, for some $b \in [1, n]$,

$$F_{\mathrm{GE}} = \Omega\left(n \cdot b^2\right), \qquad W_{\mathrm{GE}} = \Omega\left(n \cdot b\right), \qquad S_{\mathrm{GE}} = \Omega\left(n/b\right),$$

and furthermore, $F_{\mathrm{GE}} \cdot S_{\mathrm{GE}}^2 = \Omega\left(n^3\right), \quad W_{\mathrm{GE}} \cdot S_{\mathrm{GE}} = \Omega\left(n^2\right).$

Proof.

Proof by showing that $G_{\mathrm{GE}}(n)$ is a (b^2, b^3)-**path-expander** about the path corresponding to the calculation of the diagonal elements of $\mathbf{L}$.

□

Attainability

The lower bounds are attainable for Cholesky and similar costs are achievable for QR and the symmetric eigenproblem

- Tiskin's non-pivoted recursive LU and pairwise-pivoted BSP algorithms
- 2.5D LU algorithm
- $W_{\mathrm{GE}} = n^2/\sqrt{cp}$ bandwidth cost $S_{\mathrm{GE}} = \sqrt{cp}$ synchronization cost

Krylov subspace methods

We consider the *s*-step Krylov subspace basis computation

$$\mathbf{x}^{(l)} = \mathbf{A} \cdot \mathbf{x}^{(l-1)},$$

for $l \in \{1, \ldots, s\}$ where the graph of the symmetric sparse matrix $\mathbf{A}$ is a $(2m+1)^d$-point stencil.

Cost tradeoffs for Krylov subspace methods on stencils

Theorem

Any parallel execution of an s-step Krylov subspace basis computation for a $(2m+1)^d$-point stencil, requires the following computational, bandwidth, and latency costs for some $b \in \{1, \ldots s\}$,

$$F_{\mathrm{Kr}} = \Omega\left(m^d \cdot b^d \cdot s\right), W_{\mathrm{Kr}} = \Omega\left(m^d \cdot b^{d-1} \cdot s\right), S_{\mathrm{Kr}} = \Omega\left(s/b\right).$$

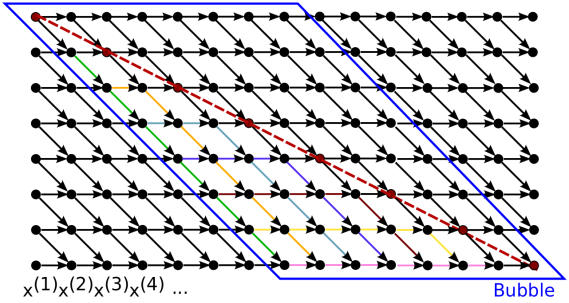
and furthermore,

$$F_{\mathrm{Kr}} \cdot S_{\mathrm{Kr}}^d = \Omega\left(m^d \cdot s^{d+1}\right), \quad W_{\mathrm{Kr}} \cdot S_{\mathrm{Kr}}^{d-1} = \Omega\left(m^d \cdot s^d\right).$$

Proof of tradeoffs for Krylov subspace methods

Proof.

Done by showing that the dependency graph of a s-step $(2m+1)^d$-point stencil is a $(m^d b^d, m^d b^{d+1})$-**path-expander**. □



Attainability

The lower bounds may be attained via communication-avoiding *s*-step algorithms (PA1 in Demmel, Hoemmen, Mohiyuddin, and Yelick 2007)

$$F_{\mathrm{Kr}} = O\left(m^d \cdot b^d \cdot s\right), W_{\mathrm{Kr}} = O\left(m^d \cdot b^{d-1} \cdot s\right), S_{\mathrm{Kr}} = O\left(s/b\right),$$

under the assumption $n/p^{1/d} = O(bm)$.

All-pairs shortest-paths problem

Given a weighted graph $G = (V, E)$ with n vertices and a corresponding adjacency matrix $\mathbf{A}$, we seek to find the shortest paths between all pairs of vertices in G

- seek the closure, $\mathbf{A}^*$, of $\mathbf{A}$ over the tropical semiring
  - $c = c \oplus a \otimes b$ on the tropical semiring implies $c = \min(c, a + b)$
  - the identity matrix $\mathbf{I}$ on the tropical semiring is 0 on the diagonal and ∞ everywhere else
  - $$\mathbf{A}^* = \mathbf{I} \oplus \mathbf{A} \oplus \mathbf{A}^2 \oplus \ldots \oplus \mathbf{A}^n = (\mathbf{I} \oplus \mathbf{A})^n$$
- numerical computation on the sum-product semiring can be computed by Gauss-Jordan Elimination

$$\mathbf{A}^* = (\mathbf{I} - \mathbf{A})^{-1}$$

- on the tropical semiring it is commonly computed by the Floyd-Warshall algorithm

Floyd-Warshall algorithm

Compute shortest paths between each pair of vertices using intermediate nodes $\{1, 2, \ldots k\}$,

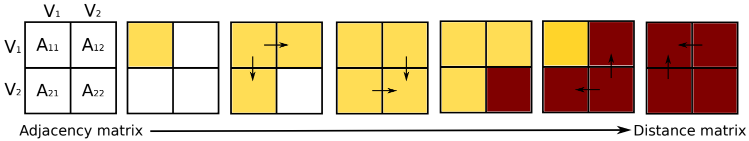
```
$D = \textsc{Floyd-Warshall}(\mathbf{A}, n)$
    $\mathbf{D} = \mathbf{A}$
    for $k = 1$ to $n$
        for $i = 1$ to $n$
            for $j = 1$ to $n$
                $d_{ij} = \min(d_{ij}, d_{ik} + d_{kj})$
```

Gauss-Jordan elimination (Floyd Warshall algorithm)

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \rightarrow \begin{bmatrix} A_{11}^* & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \rightarrow \begin{bmatrix} A_{11}^* & A_{11}^* A_{12} \\ A_{21} A_{11}^* & A_{22} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} A_{11}^* & A_{11}^* A_{12} \\ A_{21} A_{11}^* & A_{22} \oplus A_{21} A_{11}^* A_{12} \end{bmatrix} = \mathbf{B}$$

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \rightarrow \begin{bmatrix} B_{11} \oplus B_{12} B_{22}^* B_{21} & B_{12} B_{22}^* \\ B_{22}^* B_{21} & B_{22}^* \end{bmatrix} = \mathbf{A}^*$$



Parallel costs of Gauss-Jordan elimination

The floating point cost of Gauss-Jordan elimination is $F = \Theta(n^3/p)$. Our lower bounds may be applied since the computation has the same structure as Gaussian Elimination, so

$$F \cdot S^2 = \Omega(n^3), \quad W \cdot S = \Omega(n^2).$$

These costs are achieved for $W = O(n^2/p^{2/3})$ by schedules in

- Aggarwal, Chandra, and Snir 1990
- Tiskin 2007
- Solomonik, Buluc, and Demmel 2012

Lower synchronization cost via path doubling

We can compute the tropical semiring closure

$$\mathbf{A}^* = \mathbf{I} \oplus \mathbf{A} \oplus \mathbf{A}^2 \oplus \ldots \oplus \mathbf{A}^n = (\mathbf{I} \oplus \mathbf{A})^n,$$

directly via repeated squaring (path-doubling)

$$(\mathbf{I} \oplus \mathbf{A})^{2k} = (\mathbf{I} \oplus \mathbf{A})^k \otimes (\mathbf{I} \oplus \mathbf{A})^k$$

with a total of $\log(n)$ matrix-matrix multiplications, with

$$F = O(n^3 \log(n)/p)$$

operations and $O(\log(n))$ synchronizations, which can be less than the $O(p^{1/2})$ required by Floyd-Warshall.

Tiskin's path doubling algorithm

Tiskin gives a way to do path-doubling in $F = O(n^3/p)$ operations. We can partition each $\mathbf{A}^k$ by path size (number of edges)

$$\mathbf{A}^k = \mathbf{I} \oplus \mathbf{A}^k(1) \oplus \mathbf{A}^k(2) \oplus \ldots \oplus \mathbf{A}^k(k)$$

where each $\mathbf{A}^k(l)$ contains the shortest paths of up to $k \geq l$ edges, which have exactly l edges. We can see that

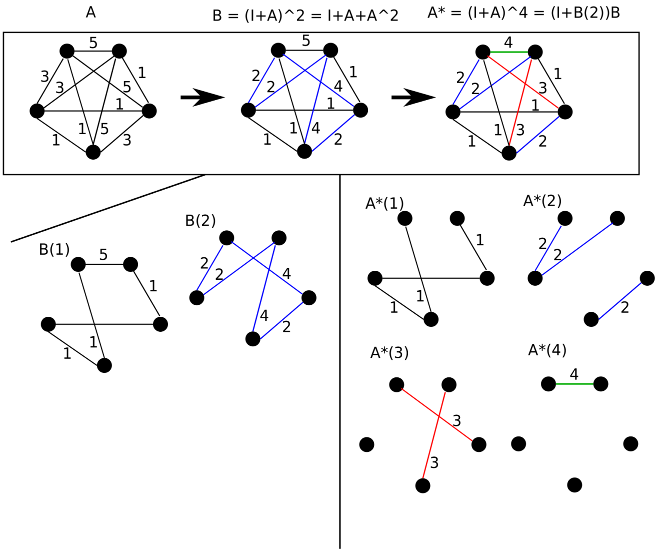
$$\mathbf{A}^l(l) \leq \mathbf{A}^{l+1}(l) \leq \ldots \leq \mathbf{A}^n(l) = \mathbf{A}^*(l),$$

in particular $\mathbf{A}^*(l)$ corresponds to a sparse subset of $\mathbf{A}^l(l)$.
The algorithm works by picking $l \in [k/2, k]$ and computing

$$(\mathbf{I} \oplus \mathbf{A})^{3k/2} \leq (\mathbf{I} \oplus \mathbf{A}^k(l)) \otimes \mathbf{A}^k,$$

which finds all paths of size up to $3k/2$ by taking all paths of size exactly $l \geq k/2$ followed by all paths of size up to k.

Path-doubling (Tiskin's algorithm)



Path-doubling

Earlier caveat:

$$(\mathbf{I} \oplus \mathbf{A})^{3k/2} \leq (\mathbf{I} \oplus \mathbf{A}^k(l)) \otimes \mathbf{A}^k,$$

does not hold in general. The fundamental property used by the algorithm is really

$$\mathbf{A}^*(l) \otimes \mathbf{A}^*(k) = \mathbf{A}^*(l+k).$$

All shortest paths of up to any length are composable (factorizable), but not paths up to a limited length. However, the algorithm is correct because $\mathbf{A}^l \leq \mathbf{A}^k(l) \leq \mathbf{A}^*(k)$.

Cost of Tiskin's algorithm

Since the decomposition by path size is disjoint, one can pick $\mathbf{A}^k(l)$ for $l \in [k/2, k]$ to have size

$$|\mathbf{A}^k(l)| \geq 2n^2/k.$$

Each round of path doubling becomes cheaper than the previous, so the cost is dominated by the first matrix multiplication,

$$F = O(n^3/p) \quad W = O(n^2/p^{2/3}) \quad S = O(\log(n)),$$

solving the APSP problem with no $F \cdot S^2$ or $W \cdot S$ tradeoff and optimal flops.

More on Tiskin's APSP algorithms

Tiskin gives a way to lower the synchronization from $S = O(\log(n))$ to $O(\log(p))$. For nonnegative edge lengths it is straightforward

- compute $\mathbf{A}^p$ via path-doubling
- pick a small $\mathbf{A}^p(l)$ for $l \in [p/2, p]$
- replicate $\mathbf{A}^p(l)$ and compute Dijkstra's algorithm for n/p nodes with each process, obtaining $(\mathbf{A}^p(l))^*$
- compute by matrix multiplication

$$\mathbf{A}^* = (\mathbf{A}^p(l))^* \otimes \mathbf{A}^p$$

since all shortest paths are composed of a path of size that is a multiple of $l \leq p$, followed by a shortest path of size up to p

Summary and conclusion

- obtained synchronization cost lower bound for any parallel schedule of Gaussian elimination
- same technique yields cost tradeoffs for Krylov subspace methods
- on the tropical semiring these are shortest-path graph algorithms, Floyd-Warshall and Bellman-Ford
- it is possible to use a different algorithm to circumvent the tradeoffs for the all-pairs shortest-paths problem
- Open question: can one circumvent the tradeoffs in an algorithm that obtain the closure of a numerical matrix?